FIXED POINT THEORY AND ITS APPLICATION IN SOLVING LINEAR AND NON-LINEAR INTEGRAL EQUATIONS

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ABSTRACT
The fixed-point theory is quite an interesting branch of mathematics that is not only limited to pure mathematics and applied mathematics but is used everywhere in our day-to-day life. If possible, we can find a fixed point in every field of science, mathematics, and engineering. Fixed point theory can be thought of as a link between the two main fields of mathematics i.e., topology and analysis. Fixed point theory can be considered the most powerful and also one of the very important tools in mathematics. This paper will particularly aim at defining the fixed-point theory in the simplest language possible and we will also be focusing on how we used the fixed-point theory for finding the solution of Linear integral equations as well as non-linear Volterra–Fredholm Integral Equations. We will try and find a viable solution for non-linear Volterra-Fredholm integral equations of different types.

Keywords: Fixed-point theory, non-linear equations, integral equations, Volterra-Fredholm equations, analysis, topology, Banach theorem.

INTRODUCTION

A fixed point as suggested by the name is a fixed point for a curve that does not change with the given transformations. If we talk about the history of fixed-point theory it belongs to topology, one important part of mathematics that was developed at the end of the nineteenth century by Johann benedict listing but at that time it was just an idea and not complete. While in the early twenty-century the idea of topological spaces was discovered. Then by the French mathematician H. Poincare, the fixed-point theory was founded. It was not discovered that early as compared to other sections in mathematics but it is still fully developed. The study of the existence of fixed points contains numerous areas of mathematics. If we talk about analysis classical and functional analysis both are required, in topology general and algebraic topology is required and also some knowledge of operator theory can be used in the study of the existence of fixed points. Fixed points as well as fixed point theorems always played an important role when it comes to theoretical explanation of various fields developed with the help of mathematics such as mathematical economies, boundary value problems, initial value problems, approximation theory, and also the game theory and this is not all there are various other fields in which fixed point theory has played a very significant role in the development. Time and again fixed-point theory has proved its importance in the development of mathematics.
With the help of recent developments in finding fixed points more efficiently and accurately, the use of fixed points has increased immensely. Now it is not only limited to the above-mentioned fields of mathematics but is being used almost everywhere. Fixed point theory is one of the major tools in solving and finding the fixed points of linear equations, differential equations, integral equations, and also in non-linear integral solutions.
According to the fixed-point theory, we can say that any map says T is a self-mapping on Y of a topological space i.e., $T: Y \to Y$ that guarantees at least one fixed point or there can be more than one fixed point. Let us say those are the points y in Y which can be further defined as $y = Ty$. Keeping the above definition of fixed point let's consider some examples of fixed points

- Translation i.e., let's take any mapping $T(y)= y+a$ where a is not equal to zero. This mapping will never have any fixed point because for all y

$T(y) \neq y, \qquad \text{for } a \neq 0$

Which is a necessary condition for any map to have a fixed point.

- Rotation, if we consider the rotation of any plane, it will always have one fixed point only i.e., the fixed will be the point of the center of rotation of that plane.
- Mapping $y \rightarrow y^2$ defined on R→R will always have only two fixed points

For y=0 $y^2 = 0$ i.e., T(y) = y
Hence, 0 is the fixed point of this curve
Also, for y =1 $y^2 = 1$ i.e., again T(y) = y
So, 1 is also the fixed point of this curve
But there exists no other point for which T(y) = y
Hence, this curve will have two fixed points only i.e., 0 and 1

- Similarly, if we consider a mapping $y \rightarrow y^3$ defined on R→R will always have only three fixed points

For, y=0 $y^3 = 0$ i.e., T(y) = y
Hence, 0 is the fixed point of this curve
Also, for y =1 $y^3 = 1$ i.e., again T(y) = y
So, 1 is also the fixed point of this curve
Also, for y =-1 $y^3 = -1$ i.e., again T(y) = y
So, -1 is also the fixed point of this curve
But there exists no other point for which T(y) = y
Hence, this curve will have three fixed points only i.e., 0, 1 and -1

- And for mapping $y \rightarrow y$ defined on R→R

For any value of y T(y) will always be equal to y. therefore, this type of mapping will have infinitely many fixed points
Remark- In the case of finding finite fixed points, we are only able to find the maximum of 3 fixed points till now.

The first fixed point theorem was given by a Dutch mathematician, L.E.J Brouwer in 1912. It was a theorem based on algebraic topology. In this theorem of fixed point Brouwer proved that any self-mapping that is continuous say T of any closed ball of 1 unit dimension say R^n will always have at least one fixed point i.e., there will always exist a point y in Y such that T(y) = y. After the proof of Brouwer's theorem, another very important theorem was proved commonly recognized as the 'Banach fixed point theorems' and 'Banach contraction principle. In this theorem, it was proved that if the self-mapping is also a contraction mapping and is present in complete metric space then there exists a fixed point and that fixed point will be unique. Hence, with the help of the Banach contraction principle, we can easily find the existence and uniqueness of a fixed point. After the Banach contraction principle was discovered, many other mathematicians worked on it and discovered the extensions of Banach contraction principles and used this result in solving linear integral equations, differential equations, linear equations as well as non-linear integral equations. Banach contraction principle was widely used as the basis for the development of applications of Fixed-point theory.

APPLICATION OF FIXED POINT TO REAL-WORLD PROBLEMS

In the last ten decades or more, the fixed-point theory has proved itself as a necessary and very strong instrument in the study of various non-linear phenomena. If we talk about today the fixed-point theory has been used in almost all fields whether it is biology, engineering, chemistry, computer science, geometry, astronomy, game theory, and many other too. The field of fixed-point theory has been technologically advanced separately and has numerous uses and is applied to almost all real-world problems. Fixed-point theory for metric spaces became necessary for non-linear functional analysis as well as for general topology. On the other hand, fixed point theory is also applied to solving problems in communication engineering. Some other real-life uses of the fixed-point theory are seen in the solutions of chemical equations, also in genetics, the fixed-point theory is used as well in the testing of algorithms uses fixed point theory.

Application to solution of linear equations

in the case of linear equations, we already know there is several direct ways for finding the solutions of linear equations. Iterative methods can also be used for tod solutions and give more accurate results. We can apply the Banach contraction principle to find the solution to linear equations and the solution will be the fixed point.

Let's understand this application of fixed point in finding the solutions of linear equations

Let's take a system of three linear equations say

$$a_1x_1 + b_1x_2 + c_1x_3 = d_1$$
$$a_2x_1 + b_2x_2 + c_2x_3 = d_2$$
$$a_3x_1 + b_3x_2 + c_3x_3 = d_3$$

And T be a continuous self-mapping i.e., T:X→X and d (Tx, Ty) ≤ k d (x, y)

Now, from the above equations, we can find the values of $x_1,\ x_2,\ x_3$.

so, $x_1 = (1 - a_1)x_1 - b_1x_2 - c_1x_3 + d_1$

$$x_2 = (1 - b_2)x_2 - a_2x_1 - c_2x_3 + d_2$$
$$x_3 = (1 - c_3)x_3 - b_3x_2 - a_3x_1 + d_3$$

Using the knocker delta, the above system of equations will be equivalent to$x_i = \sum \alpha_{ij}x_j + d_j$

We can see the above representation of given system of linear equations is of the form

x = Ax + d

we know every linear mapping is continuous

so, T(x) = Ax + d is continuous

d (Tx, Ty) = |Tx – Ty| = |Ax – Ay| = |A| |x-y| = |A| d (x, y)

Therefore, T is contraction if |A| < 1

Here we have taken an example of a system of three linear equations but the above result is also true for system of linear equations having any number of equations.

Hence, by using Banach contraction principle we can say that any system of linear equations having n equations will have a unique fixed point for $|\alpha_{ij}| \leq 1$

Application to Solution of non-linear Volterra-Fredholm integral equation by fixed point method:

In this we will take a set P: =C ([x, y], R); of continuous real value function which is define in [x, y]. Now after taking the set P, we should define

δ_e: T× T→R and $\gamma: T \times T \to [1, \infty)$

by: δ_e(a, b) = sup |a(z) +b(z)|^2, z∈[x, y] and $\gamma(a, b)$=| a(z) |+| b(z)| +1

From the above equation, we can see clearly that (T,δ_e) is fully extended b-metric space.

Now we will take the non-linear Volterra- Fredholm integral equation:

a(z) = $\lambda_1 \int_x^z c1\left(z, T, a(T)\right)ds$ + $\lambda_2 \int_x^y c2$(z, T, a(T)) ds

where x ≤z ≤y,

In the above equation a(z) stands for unknown solution, c_i (z, T, a(z)); (i =1,2) stands for smooth function and λ_i, x, and y are constants.

For the above non-linear Volterra- Fredholm integral equation we should assume x=0 for an easy thorough inspection.

Let us take the following conditions are true:

1. The mapping f: C [0, y]→C [0, y] defined by:

$$f(a(z)) = \lambda_1 \int_0^z c1\left(z, T, a(T)\right)ds + \lambda_2 \int_0^y c2\left(z, T, a(T)\right)ds;$$

for every a ∈C [0, y] where T, z, y are related as 0 ≤ T≤ y

we can clearly see that the above mapping f is a continuous mapping and the c_i will be defined as:

c_i: [0, y] ×[0, y] × R→R

2. For b: [0,∞) → [1,∞) where we take b(z)<z for every z>0.

3. Now let us take c_i for some constant A_i which should satisfy the below equation:

|c_i (z, T, a(T)) – c_i (z, T, b(T)) | ≤ A_i [Y| a(T)-b(T)|)]$^{r1/2}$,

Where T, z, b are related as 0≤ T≤ z≤ b and i = 1,2.

4. Now after all the above three conditions let's assume $\alpha z + \beta y < 1$, where α represents $\lambda_1 A_1$ and β represents $\lambda_2 A_2$.
As we have already assumed that all the above conditions are true and hence, we can say that the above non-linear Volterra –Fredholm equation has only one possible solution.
Take:

$|fa(z) - fy(z)|2 = |\lambda 1\int_0^z c1\left(z,T,a(T)\right)ds + \lambda 2\int_0^y c2\left(z,T,a(T)\right)ds - (\lambda_1\int_0^z c1\left(z,T,b(T)\right)ds + \lambda_2\int_0^y c2\left(z,T,b(T)\right)ds)|^2$

$= |\lambda_1[\int_0^z c1\left(z,T,a(T)\right) - c1(z,T,b(T)))ds] + \lambda_2[\int_0^y c2\left(z,T,a(T)\right) - c2(z,T,b(T)))ds]|^2$

$\leq |\lambda_1 A_1(Y\,|\,a(T) - b(T)|)^{r1/2} z + \lambda_2 A_2(y\,|\,x(T) - b(T)|)^{r1/2} y|^2$

$= |(\alpha z + \beta)(Y|a(T) - b(T)|)^{r1/2}|^2$

$< (Y(|a(T) - b(T)|)^{r1}$

$= (Y\,(\delta_e(a,b))^{r1}$

This gives,

$\underbrace{sup}_{z\epsilon[0,y]} |fU(z) - fV(z)|^2 \leq (Y\delta_e(a(z),y(z)))^{r1},$

This gives us,

$\delta_e(f\left(a(z)\right), f\left(y(z)\right) \leq \left(Y\delta_e\left(a(t),b(z)\right)\right)^{r1}$, for every a, b ϵT.

Which will provide us,

$Y\,(\delta_e\left(f\left(a(z)\right), f\left(b(z)\right)\right) \leq \delta_e\left(f\left(a(z)\right), f\left(b(z)\right)\right)$

$= (Y\delta_e\left(a(z)\right), b(z)))^{r1}$

We can say that f completes all the necessary conditions of the Banach Contraction Principal corollary.
Therefore, we can say that f will have a unique fixed point. This can also be understood as that there exists a single solution for the above-mentioned integral equation that is non-linear Volterra-Fredholm equation.

CONCLUSION

As we have already discussed in the above paper fixed point theory has many applications in all the fields of sciences, engineering, and mathematics. If we talk about today, we cannot imagine working without fixed point theory as it has become the most important and efficient tool to work on each and every space. The fixed point is a combination of analysis and topology and hence provides an upper hand. Initially, everyone thought fixed point theory could only be used for linear problems, but with the further developments, it was also used for non-linear integral equations. This paper mostly deals with the non-integral Volterra Fredholm integral equations. But fixed-point theory is not limited to it only. As fixed point is the already a fully developed topic hence, it has many important theorems proved by mathematicians. Not only the solution of integral solutions but also the solutions of differential equations can be found with the help of fixed point.

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